# Can a Pythagorean Triangle Repeat Its Gravitational Dance?

Exact Reductions and Partial Results toward the
Pythagorean–Burrau Nonperiodicity Conjecture

Working research draft

September 5, 2026

**Abstract**

Release three Newtonian point masses from rest at the vertices of an integer right triangle, with each mass equal to the length of its opposite side. Must the resulting motion fail to be collision-free and periodic? We review the historical experiment, report direct counterexample searches, and develop partial results toward this conjecture. Exact scaling reduces the integer family to a rational curve. Time reversal reduces periodicity to a second simultaneous stop; the Lagrange–Jacobi identity restricts such a stop to a strict maximum of the moment of inertia. Validated flow covers and a terminal escape estimate exclude periodicity on a closed interval near $u = 0.29$. Separate endpoint arguments give a punctured near-isosceles interval and infinitely many thin-triangle escape windows, including a positive-density subset of an explicit primitive family. Numerical continuation finds nearby periodic candidates in a larger mass–side family, none satisfying the right-angle equation. The universal conjecture remains open. We identify a missing terminal bound in an earlier torque-history argument and retain the resulting conditional reduction with that hypothesis explicit.

## 1 A triangle released from rest

Place three small, perfectly concentrated masses at the corners of a right triangle. Give the triangle sides of length 3, 4, and 5, and give the bodies masses 3, 4, and 5, placing each body opposite the side with its own number. Let go of all three at once. There is no initial push, no friction, and no force except their mutual Newtonian gravity. What happens?

At first the bodies fall toward one another. Later they can exchange partners, pass extremely close, or fling one another far away. A particularly appealing possibility is that they eventually come back to their starting positions, stop together, and repeat the same dance forever. This paper asks whether that can happen for *any* integer right triangle when its masses and opposite sides are tied together in this way. The conjecture is that it cannot. We have substantial partial results, but no proof of the universal claim and no counterexample.

### 1.1 From Meissel's question to computer experiments

The starting point is a conversation, rather than a known published conjecture. In his 1913 paper, Carl Burrau recalled speaking in 1893 with the Kiel mathematician Ernst Meissel, who believed that the 3–4–5 initial conditions would produce periodic motion. Burrau also wanted to test variable-step numerical integration on a demanding example. His original account credits Sigurd Kristensen with a substantial part of the calculation [1]. The attractive integers made the experiment easy to describe; they did not make its motion easy to calculate.

The decisive revival preceded the famous 1967 publications. Szebehely's own account records calculations in 1966 by Standish at Yale, Spinelli with Lecar and Szebehely at NASA's Institute for Space Studies, and Stanek under Stiefel at ETH Zürich. It says that Szebehely suggested the revival, but does not say how he first encountered Burrau's paper. We have found no primary-source statement identifying a person or occasion that brought it to his attention [2].

Nor should the intervening years be described as a complete halt in work on three bodies of comparable masses. Szebehely discusses Strömgren's periodic-orbit work, Zumkley's 1941 integration, and Garcia's 1966 extension of Zumkley's example [2]. Zumkley explicitly cites Burrau on the first page of his paper, but uses three unit masses and a slide rule [4]. Strömgren's 1919 paper likewise studies a different periodic-orbit construction [3]. These are related problems, not evidence that the exact 3–4–5 trajectory was continuously recomputed. The sources examined here do not establish a continuation of that exact calculation between Burrau's work and the documented 1966 revival. This distinction is more defensible than a claim of fifty-four years of complete silence.

Szebehely and C. Frederick Peters followed the original motion far enough to identify its eventual behavior: the two heavier bodies, of masses 4 and 5, form a binary, while the mass-3 body escapes. Their paper explains that Burrau had stopped at $t = 3.35$, before this outcome was apparent. Their numerical resolution refuted Meissel's expectation for that particular triangle [5]. It did not settle every integer right triangle. In a separate paper that year, they found a nearby periodic solution with a binary collision and perturbed initial distances [6]. That example is outside the collision-free question considered here.

### 1.2 The modern question is more specific than free fall

Starting from rest does not itself rule out periodic motion. Standish's collisionless periodic example appeared in 1970 [7]. Kiyotaka Tanikawa, Hiroaki Umehara, and Hiroshi Abe developed systematic searches for binary- and triple-collision orbits in 1995; Tanikawa's 2000 continuation studied the structure of this free-fall initial-condition space [8, 9]. Their work makes collisions part of the organization of the problem, rather than merely numerical inconveniences. Moeckel, Montgomery, and Venturelli subsequently developed the rigorous passage from a configuration at rest to the first collinear configuration, and constructed periodic collision orbits in the isosceles problem [10]. Montgomery's *Dropping Bodies* gives a modern account of periodic free-fall motion and its reversal symmetry [11]. None of these statements identifies the all-integer nonperiodicity claim of this paper as a conjecture due to Tanikawa or Montgomery.

Recent computations make the distinction still more important. Li and Liao's 2019 catalogue reports 316 periodic free-fall solutions, including 313 new examples, for several mass ratios [12]. Hristov, Hristova, Dmitrašinović, and Tanikawa's 2024 equal-mass search reports 12,409 distinct solutions below its specified scale-invariant period cutoff [13]. These catalogues vary initial shapes more freely than our experiment. Equal masses cannot satisfy our right-triangle mass–side rule. Unequal-mass entries are useful starting guesses, but must still satisfy both the mass–side relation and the right-angle equation.

The original Pythagorean problem also remains a numerical benchmark: Rantala, Pihajoki, Mannerkoski, Johansson, and Naab use it to test their 2020 regularized integrator [14]. Chitan, Mylläri, and Haque revisited Pythagorean black-hole triples in 2022, and Chitan, Mylläri, and Valtonen studied spin effects in 2025 [15, 16]. Those relativistic extensions address encounters, escape, and mergers under modified equations. They demonstrate continued interest in the experiment, while leaving the exact Newtonian conjecture here open.

### 1.3 One number specifies the experiment

The rule "mass equals opposite side" is understood in chosen mass and length units, with gravitational constant set to one. Enlarging all three masses and all three lengths by the same factor changes the clock rate, but preserves whether the motion repeats. We can therefore divide by the hypotenuse and work with a longest side and largest mass equal to one. Write the two other side lengths, and their corresponding masses, as $A$ and $B$. The right-angle condition becomes $A^2 + B^2 = 1$.

A convenient way to describe every such triangle is to slide a single parameter $u$ through the interval $(0, 1)$:

$$A = \frac{1-u^2}{1+u^2}, \qquad B = \frac{2u}{1+u^2}. \tag{1}$$

Geometrically, if $(A, B) = (\cos\theta, \sin\theta)$, then $u = \tan(\theta/2)$. No trigonometry is required to use the rational formulas. If $u = p/q$ is a fraction in lowest terms, multiplying through by $q^2 + p^2$ gives Euclid's integer triple

$$(q^2 - p^2,\ 2pq,\ q^2 + p^2).$$

Divide the triple by two when both $p$ and $q$ are odd. Conversely, every primitive integer right triangle arises this way, allowing exchange of its two legs. "Primitive" means that the three integers have no common factor. For example, $u = 1/2$ gives $(3, 4, 5)$, and $u = 1/3$ gives $(4, 3, 5)$ after reduction. These are the same experiment with the first two labels exchanged.

To avoid counting it twice, we use

$$0 < u \leq u_{\text{iso}} := \sqrt{2} - 1, \qquad A \geq B.$$

Near zero the triangle is very thin; at the other endpoint its two legs are equal. That equal-leg endpoint has irrational $u$, so it belongs to the continuous family used in the proof, but not to the integer conjecture. Changing $u$ changes the masses *and* the shape together. It does not mean moving one body while keeping three arbitrary masses fixed.

### 1.4 What would count as a counterexample?

A repeating picture is not enough. Each labelled body must return to its own position with its own velocity in the same fixed coordinate frame, without any collision along the way. A return after rotating the picture, swapping bodies, or mathematically continuing through a collision does not meet this definition. A numerical near-return, however precise, is a candidate to investigate rather than an exact periodic orbit.

There is a useful simplification. Newton's equations work equally well backward in time. If a collision-free trajectory that starts at rest ever stops *all three bodies simultaneously* a second time, it must retrace its entire path and return to the beginning. Conversely, every periodic trajectory starting at rest has such a second stop. Mathematicians call a simultaneous stop a *brake*. The conjecture can therefore be phrased as: *no integer Pythagorean experiment has a second brake before its first collision.* Section 2 describes our direct attempts to find one.

### 1.5 What the proof attempt achieves

The difficulty is to control every future encounter for infinitely many triangles. Figure 2 shows how nearby rational parameters can produce visibly similar motion over a finite time, while close approaches make numerical errors and small changes of parameter matter. A dense plot is not a proof that a whole interval has been covered.

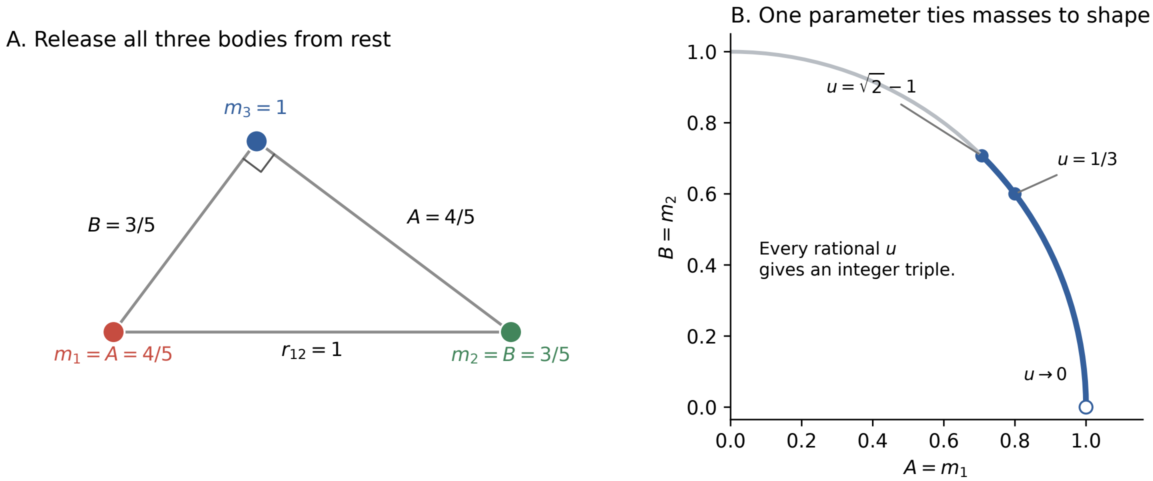


Figure 1: The experiment and its parameter. Left: the normalized $u = 1/3$ triangle, with masses $(4/5, 3/5, 1)$ opposite equal-numbered side lengths; all velocities initially vanish. Right: the allowed mass pairs lie on the quarter unit circle. The darker arc is the fundamental interval $0 < u \leq \sqrt{2}-1$; the thin-triangle endpoint $u = 0$ is excluded. Rational $u$, rather than arbitrary points selected numerically on the arc, produces the integer triples.

Three kinds of progress are established by the arguments and certificates assembled here. First, an exact reduction identifies the only moments at which a second brake is possible: strict maxima of the system's moment of inertia, a measure of its overall spread. At such a maximum the changing triangle still has to stop changing shape. Second, validated integration and a forward escape estimate exclude every second brake on the explicit real interval $[0.29, 0.2900000101]$. Third, endpoint arguments give a punctured neighborhood of the equal-leg triangle and infinitely many open intervals accumulating at the thin-triangle end. On the explicit family $(4n^2 - 1, 4n, 4n^2 + 1)$, the latter arguments cover a set of indices of positive lower density: a positive fraction in the limiting lower-density sense, not necessarily every large index.

These results leave much of the parameter interval unclassified. The near-equal-leg cutoff is existential, and the thin-triangle result covers escape windows rather than all phases. We also develop torque-history identities that could help control the remaining region. The scalar sign-change shortcut proposed in an earlier draft needs an additional endpoint bound; Section 7 explains the correction. The paper's outcome is a set of rigorous reductions and partial exclusions, with the remaining global obligation stated explicitly.

## 2 Attempts to construct a counterexample

The search starts from an adversarial question: can a known periodic free-fall orbit be adjusted until it lies exactly on the Pythagorean curve? This is more directed than waiting for a long trajectory to nearly return. It also tests whether the apparent scarcity of second brakes is merely an artifact of sampling.

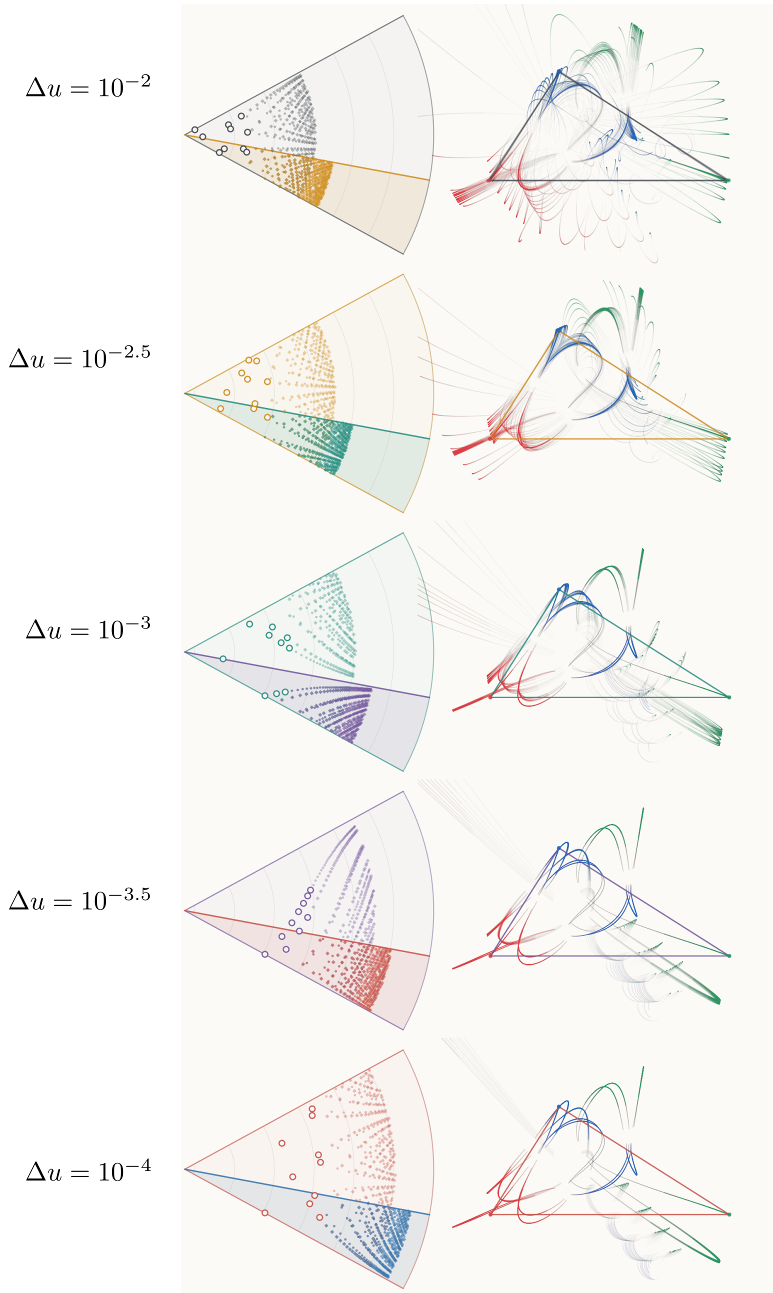


Figure 2: Progressively narrower collections of integer experiments near $u = 0.29$, using the project's original fan–trajectory visualization. From top to bottom the intervals are $[0.29, 0.29 + \Delta u]$, with $\Delta u = 10^{-2}, 10^{-2.5}, 10^{-3}, 10^{-3.5}, 10^{-4}$. Each left panel displays 400 primitive triples of smallest hypotenuse in its interval, with radius on a common affine $\log_{10} c$ scale and its angular interval expanded to one display radian. The highlighted sector is the next narrower interval; large rings select ten triples whose trajectories appear at right. Red, green, and blue track the three bodies in the source plot's leg ordering. Right panels share one spatial scale, show normalized physical times $0 \leq t \leq 4$, and clip outgoing paths. Opacity decreases with body speed; it is not an integration-error measure. Even the narrowest displayed interval is about 9,901 times wider than the certified interval. This ordinary numerical illustration conveys the search geometry, not validated parameter coverage.

| Seed | $m_1$ | $m_2$ | Second-brake time | $D$ |
|---|---|---|---|---|
| $F_1$ | .536940689 | .839991651 | 1.337970014 | $-.006108723$ |
| $F_2$ | .485763526 | .878297063 | 1.413330015 | $+.007371934$ |
| $F_3$ | .343754597 | .887462426 | 1.074865097 | $-.094243220$ |
| $F_4$ | .335421060 | .903827720 | 1.133475174 | $-.070588165$ |
| $F_5$ | .474742622 | .896107795 | 1.544076262 | $+.028389737$ |
| $F_{30}$ | .594811647 | .801774973 | 6.289233824 | $-.003355997$ |

Table 1: Ordinary numerical periodic candidates in the larger mass–opposite-side family. None is Pythagorean. Digits are displayed for reproducibility, not as interval-certified enclosures. The flight time is physical normalized time; twice this value would be a period at an exact collision-free second brake. Seed names refer to catalogue origins, not to established connected branches of solutions.

### 2.1 Impose the geometric constraints in stages

Normalize $m_3 = r_{12} = 1$. In the larger family where each mass equals its opposite side, a valid nondegenerate triangle has

$$q_1 = (-1/2, 0), \quad q_2 = (1/2, 0), \quad q_3 = (x, y), \quad x = \frac{m_2^2 - m_1^2}{2}, \quad y = \sqrt{m_2^2 - (x + 1/2)^2} > 0. \tag{2}$$

There are now two adjustable mass ratios. Periodic candidates in this larger family need not be right triangles. Their remaining defect is

$$D = m_1^2 + m_2^2 - 1. \tag{3}$$

An exact zero of $D$ is required even to challenge the stronger conjecture for real parameters. An integer counterexample additionally needs rational $u = m_2/(1 + m_1)$.

Our September 2026 campaign starts from six entries of Li and Liao's unequal-mass catalogue [12]. In the numerical construction we solve three independent second-brake equations together with two mass–side matching equations, varying the two initial-position coordinates, two mass ratios, and a regularized flight time. Levi–Civita coordinates replace the selected pair separation by a complex square and slow the clock near close approaches. An atlas of pair choices avoids relying on one coordinate chart for every encounter. Analytic variational equations supply the shooting derivatives; pseudo-arclength continuation recovers seeds for which direct correction stalls.

### 2.2 What the numerical checks establish

The corrected shooting residuals are at most about $1.2 \times 10^{-12}$. Reintegration with an implicit Radau method, comparison with DOP853, and backward reversal reproduce the candidate states to roughly $2 \times 10^{-12}$. The corresponding five-equation Jacobians have ordinary smallest singular values between approximately 0.199 and 0.870 in the recorded physical-parameter convention. Their numerical nonsingularity suggests isolated solutions of the five matching equations, rather than freely adjustable families that automatically cross $D = 0$. It does not prove existence or isolation: those conclusions require a validated implicit-function calculation.

The closest sampled separations range from about $3.4 \times 10^{-6}$ to $5.0 \times 10^{-4}$. These values explain the use of regularization and also limit the evidence: sampling alone does not certify positive separation for every intervening time. Reconstructed energy is especially sensitive to cancellation near the deepest passages; the campaign's worst sampled physical energy error is about $2.5 \times 10^{-7}$.

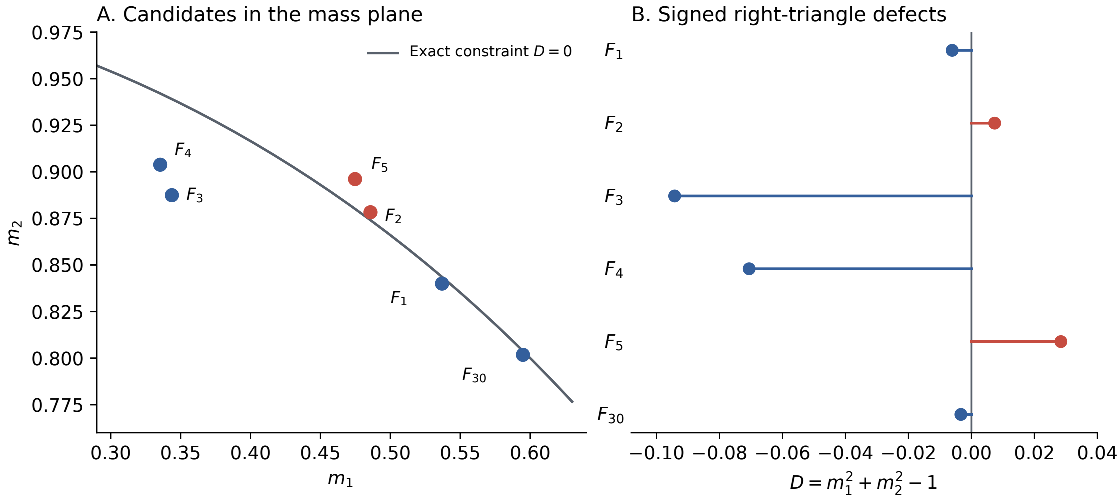

Figure 3: The six numerical candidates of Table 1 and exact right-triangle constraint. The left panel places them relative to $m_1^2 + m_2^2 = 1$; the right displays each signed defect directly. Opposite signs at unrelated candidates are not an intermediate-value bracket. No connecting solution branch is established by these points.

Small shooting residuals must not be presented as equally small error bounds for every physical observable.

We separately impose the exact right-triangle parameterization and look for small kinetic energy at local maxima of the moment of inertia. Four local searches seeded from $F_1, F_2, F_3, F_5$ converge to positive minima, the smallest about $1.28 \times 10^{-4}$. The $F_{30}$ search reaches its evaluation limit and remains noise-limited; the $F_4$ event correction leaves the required strict-maximum branch. Neither incomplete search is an exclusion theorem. No exact right-triangle second brake was found, so there is no candidate on which rational reconstruction would be warranted. Machine-readable results and exact replay commands accompany the campaign report listed in Appendix A.

### 2.3 Two tempting continuation arguments that do not suffice

The defects of $F_1$ and $F_2$ have opposite signs. This would be useful if we had a connected collision-free periodic branch joining them, on which $D$ varied continuously. We do not. Their labelled half-orbit syzygy sequences are different, and at zero angular momentum a noncollinear collision-free orbit crosses every syzygy transversely. Consequently those finite sequences cannot change on a connected finite-flight-time brake family without an excluded event or endpoint. Opposite defects alone are not a bracket.

Nor does a rank deficiency automatically construct the missing branch. For five smooth equations in five unknowns, a rank-four Jacobian leaves one direction to inspect, but the remaining scalar equation after Lyapunov–Schmidt reduction can be $s^2 = 0$, with an isolated root. A branch requires that residual equation to vanish along the proposed continuation. Similarly, raw free-group words from the catalogue do not distinguish these brake loops: a full brake orbit retraces its path and its loop in collision-free shape space is null-homotopic. The ordered, labelled syzygy sequence is the useful finite-itinerary information here.

The search has therefore produced well-tested near misses and sharper requirements for a counterexample. It has not supplied a probability of nonexistence, a numerical proof of the conjecture, or an obstruction to unsearched long-period families.

# 3 The exact problem and its reductions

Let $a^2 + b^2 = c^2$ with positive integers, set $(m_1, m_2, m_3) = (a, b, c)$, and put

$$q_1 = (-c/2, 0), \quad q_2 = (c/2, 0), \quad q_3 = ((b^2 - a^2)/(2c), ab/c), \qquad \dot{q}_i = 0.$$

The side opposite each mass has the same length as that mass. We work with $G = 1$ and the equations

$$\ddot{q}_i = \sum_{j \neq i} m_j \frac{q_j - q_i}{|q_j - q_i|^3}.$$

The conjecture asserts that no maximal classical solution has a collision-free labelled period. Returns modulo rotation or label exchange and regularized continuation after collision are excluded.

## 3.1 Scaling and the rational parameter curve

**Lemma 1** (Simultaneous mass–length scaling)**.** *If $q_i(t)$ solves the equations with masses $m_i$, then for $k > 0$*

$$\widetilde{m}_i = k m_i, \qquad \widetilde{q}_i(t) = k q_i(t/k)$$

*solves the rescaled problem. A labelled period $T$ corresponds exactly to a period $kT$.*

*Proof.* Both sides of Newton's acceleration equation acquire the factor $k^{-1}$. Velocity is unchanged after the corresponding time substitution, and mutual distances acquire the positive factor $k$. ☐

It follows that only primitive triples matter. For $u \in (0, 1)$ define

$$A(u) = \frac{1 - u^2}{1 + u^2}, \qquad B(u) = \frac{2u}{1 + u^2}.$$

If $u = p/q$ is reduced, Euclid's integer triple is $(q^2 - p^2, 2pq, q^2 + p^2)$ and its common divisor is two exactly when $p, q$ are both odd. Conversely Euclid's theorem recovers every ordered primitive triple. The inverse is $u = B/(1 + A)$, and leg exchange is the involution $u \mapsto (1 - u)/(1 + u)$. Its fixed point is $\sqrt{2} - 1$, giving the fundamental interval $0 < u \leq \sqrt{2} - 1$ for the real family.

After normalizing $c = 1$, masses and positions are

$$(A, B, 1), \qquad q_1 = (-1/2, 0), \quad q_2 = (1/2, 0), \quad q_3 = ((B^2 - A^2)/2, AB).$$

## 3.2 Second brakes and a residual valid at collinearity

**Lemma 2** (Second-brake lemma)**.** *A classical free-fall segment generates a collision-free labelled periodic solution if and only if all inertial velocities vanish at some positive time before collision.*

*Proof.* Uniqueness and time reversal give $q(-t) = q(t)$. A brake at $\tau$ gives $q(\tau + s) = q(\tau - s)$ and hence a labelled period $2\tau$. Conversely, a labelled period $T$ combined with the symmetry at zero gives reflection symmetry about $T/2$, and therefore zero velocity there. The period $2\tau$ need not be minimal unless $\tau$ is the first positive brake. ☐

Write $r_{ij} = |q_j - q_i|$, $K = \frac{1}{2} \sum_i m_i |\dot{q}_i|^2$, $U = \sum_{i<j} m_i m_j / r_{ij}$, and $H = K - U = -U_0$. In the center-of-mass frame define reduced masses $\mu_1 = m_1 m_2/(m_1 + m_2)$ and $\mu_2 = m_3(m_1 + m_2)/(m_1 + m_2 + m_3)$, and Jacobi vectors

$$X = q_2 - q_1, \qquad Y = q_3 - \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$$

and Hopf invariants

$$h(X,Y)=\left(\frac{|X|^2-|Y|^2}{2},X\cdot Y,X\times Y\right).$$

We use the quotient residual

$$\mathcal{B}(u,t)=\frac{d}{dt}h(X(u,t),Y(u,t)).$$

Away from triple collision, $Dh$ has rank three and kernel equal to the common rotation direction. Since total linear and angular momentum vanish, $\mathcal{B}=0$ is equivalent to vanishing of every labelled inertial velocity. Unlike three mutual-distance derivatives, this remains true at syzygy.

# 4 From an infinite trajectory to a finite certificate

Let

$$I=\mu_1|X|^2+\mu_2|Y|^2,\qquad \sigma=\mu_1\overline{X}\dot{X}+\mu_2\overline{Y}\dot{Y},\qquad \zeta=\mu_1\overline{X}\dot{X}-\mu_2\overline{Y}\dot{Y}.$$

Here complex multiplication identifies the oriented plane with $\mathbb{C}$. Then $\dot{I}=2\operatorname{Re}\sigma$ and the angular momentum is $L=\operatorname{Im}\sigma$. Consequently, at $L=0$ and away from $Y=0$,

$$\dot{I}=0\ \text{ and }\ \zeta=0\quad\Longleftrightarrow\quad \dot{X}=\dot{Y}=0.$$

The implication from a brake to these three scalar equalities is unconditional. At the Jacobi degeneracy $Y=0$, the Hopf residual $\mathcal{B}$ above supplies the coordinate-independent converse.

This splitting has an exact shape interpretation. Put $J_X=\mu_1|X|^2$, $J_Y=\mu_2|Y|^2$, $s=J_X/I$, and $\phi=\arg Y-\arg X$. At every zero of $\dot{I}$ with $X,Y\neq 0$ and $L=0$,

$$\boxed{\zeta=I\left(\dot{s}-2is(1-s)\dot{\phi}\right),\qquad |\zeta|^2=\frac{8KJ_XJ_Y}{I}=8KI\,s(1-s).}$$

Indeed $\sigma=0$ gives $\zeta=2\mu_1\overline{X}\dot{X}=-2\mu_2\overline{Y}\dot{Y}$; decomposing each Jacobi velocity into radial and angular parts proves both formulas. Thus a maximum event is a brake exactly when its shape velocity vanishes, and the origin-avoidance problem can equivalently be written as the scalar strict inequality $K=U-U_0>0$. Because this scalar is nonnegative and has an even-order zero at a transverse brake, this reformulation does not create a sign-change argument by itself.

There is also a brake residual intrinsic to any binary Levi–Civita chart. For a selected pair write $g=q_j-q_i=w^2$, $dt=|w|^2d\sigma$, and $z=dw/d\sigma$. Let $G$ be the third body relative to the selected pair's center of mass and $P=\dot{G}$. With reduced masses $\mu_g,\mu_G$, angular momentum is

$$L=2\mu_g\operatorname{Im}(\overline{w}z)+\mu_G G\times P.$$

Thus, on a collision-free zero-angular-momentum trajectory and wherever $G\neq 0$,

$$\boxed{(z_r,z_i,G\cdot P)=0\quad\Longleftrightarrow\quad\text{a labelled brake.}}$$

Indeed $z=0$ gives $\dot{g}=0$; then $L=0$ gives $G\times P=0$, while $G\cdot P=0$ forces $P=0$. Total momentum removes the common translation. This residual stays regular through arbitrarily close selected-pair passages; at $G=0$ one changes Jacobi tree or uses $\mathcal{B}$.

**Theorem 3** (Brake-event identities)**.** *Along a collision-free free-fall solution with initial potential $U_0$, every brake satisfies*

$$K=0,\qquad U=U_0,\qquad \ddot{I}=-2U_0<0,\qquad r_{ij}\geq\frac{m_im_j}{U_0}.$$

*At any zero of $\dot{I}$ for which $\ddot{I}\geq 0$, one instead has $K\geq U_0$.*

*Proof.* With positive potential magnitude $U$ and energy $H = K - U = -U_0$, the Lagrange–Jacobi identity is

$$\ddot{I} = 4K - 2U = 2K - 2U_0 = 2U - 4U_0.$$

At a brake $K = 0$, giving the first three assertions. Since every summand $m_i m_j / r_{ij}$ is positive and bounded above by $U = U_0$, the separation bound follows. If $\ddot{I} \geq 0$, the middle expression instead gives $K \geq U_0$. $\square$

**Corollary 4** (Validated-cover criterion)**.** *Suppose a classical free-fall solution has a launch window $[0, t_L]$ on which $U < 2U_0$. Suppose validated flow enclosures cover every time in $[t_L, t_*]$, and every enclosure proves at least one of*

$$\dot{I} \neq 0, \qquad K > 0, \qquad \mathcal{B} \neq 0.$$

*Then there is no second brake in $(0, t_*]$. If the terminal enclosure satisfies a proved forward collision-or-escape criterion that excludes every later brake, the initial free-fall solution is nonperiodic.*

*Proof.* At launch $\dot{I}(0) = 0$, whereas $U < 2U_0$ implies $\ddot{I} < 0$. Consequently $\dot{I}(t) < 0$ for $0 < t \leq t_L$. The remaining disjunction excludes a brake at every covered time. This argument applies fiber by fiber when $t_*$ depends on a parameter; both section images and the complete intervening flow tubes must be covered. The initial brake itself is never subjected to the nonvanishing disjunction. $\square$

This criterion has been implemented with exact rational initial data and MPFR interval arithmetic. The archived certificates prove the conjecture for

$$(21, 20, 29),\ (5311, 5280, 7489),\ (8319, 8200, 11681),\ (33439, 32400, 46561),\ (72, 65, 97)$$

in both leg orderings. These are five point theorems, not a parameter-cover argument.

**Lemma 5** (Terminal escape certificate)**.** *For a prospective binary of total mass $M$ and an outer body of mass $m_c$, let $r, \rho$ be the inner and outer Jacobi radii and $e = |\dot{x}|^2/2 - M/r$. Choose $\eta > 0$, set $R = M/\eta$ and $s_0 = \rho_0 - R$, and suppose*

$$s_0 > 0, \quad \dot{\rho}_0 > 0, \quad \delta = \frac{1}{2}\dot{\rho}_0^2 - \frac{M + m_c}{s_0} > 0,$$

$$e_0 + \frac{m_c \sqrt{2MR}}{\sqrt{2\delta}\, s_0^2} < -\eta.$$

*Then the future solution either ends in inner binary collision or the outer body escapes with $\dot{\rho} \geq \sqrt{2\delta}$ forever. In particular there is no later classical brake.*

*Proof.* On the bootstrap region $e < -\eta$, one has $r < R$ and $r|\dot{x}| < \sqrt{2MR}$. The Newton field derivative bound $\|D(v/|v|^3)\| \leq 2/|v|^3$ gives

$$|\dot{e}| \leq \frac{2 m_c r |\dot{x}|}{(\rho - r)^3}.$$

Meanwhile

$$J = \frac{1}{2}\dot{\rho}^2 - \frac{M + m_c}{\rho - R}$$

is nondecreasing while $\dot{\rho} > 0$. Hence $\rho \geq \rho_0 + \sqrt{2\delta}(t - t_0)$, and the total future tidal increase of $e$ is at most the allowance in the hypothesis. Its strict margin closes the bootstrap. $\square$

**Theorem 6** (Validated middle first-maximum interval)**.** *For every real*

$$\frac{29}{100} \leq u \leq \frac{14501}{50000},$$

*the tied classical trajectory is collision-free through its first positive local maximum of $I$, and this maximum is not a labelled brake. The unique intervening minimum and the first subsequent maximum obey*

$$t_{\min} \in [0.758038, 0.758045], \qquad t_{\max} \in [1.32908, 1.33943].$$

*Proof.* The exact tied launch is embedded as a one-dimensional mean-value graph with $u$ retained as a frozen state coordinate. A pinned MPFR/CAPD $C^1$ Poincare calculation propagates this graph first in a pair–13 Levi–Civita chart and, after an exact Jacobi-tree transformation at $t = 1$, in a pair–23 chart. On every accepted-step enclosure, all three mutual separations are strictly positive. The launch-concavity window and whole-leg audits give $\dot{I} < 0$ after launch to the first minus-to-plus zero and $\dot{I} > 0$ from that zero to the first later plus-to-minus zero. At the minimum, $U/U_0 \in [5.17942, 5.18335]$; at the maximum, $U/U_0 \in [0.983016, 1.01907] \subset (0, 2)$, proving both events strict by Lagrange–Jacobi. In the final selected-pair chart the validated residual has

$$\operatorname{Im} z \in [-0.0500738, -0.0113187], \qquad P_x \in [-0.0561842, -0.00917817].$$

A labelled brake forces $z = P = 0$, so either strict component excludes it. The exact chart identities are symbolically regressed. Two abutting tiles cover the stated interval and the archived independent replay reproduces the left-tile terminal PASS enclosure. The source, logs, and pinned dependency hash are indexed in Appendix A. □

This theorem excludes only the first positive maximum. It does not rule out a brake on a later maximum branch and therefore is not, by itself, a nonperiodicity theorem on the displayed interval.

**Theorem 7** (Validated middle real interval)**.** *For every real*

$$\frac{29}{100} \leq u \leq \frac{2900000101}{10000000000},$$

*the tied maximal classical solution has no second labelled brake at any collision-free time and hence is not a classical periodic solution. Consequently the Pythagorean–Burrau conjecture holds for infinitely many primitive triples in this interval, including* $(9159, 5800, 10841)$ *at* $u = 29/100$.

More explicitly, for every integer $k \geq 99010$ set

$$u_k = \frac{290k + 1}{1000k} = \frac{29}{100} + \frac{1}{1000k}.$$

The numerator and denominator are coprime and of opposite parity, and the interval inequality is equivalent to $1/(1000k) \leq 101/10^{10}$. The theorem therefore proves nonperiodicity for the primitive triples

$$\left(915900k^2 - 580k - 1, 580000k^2 + 2000k, 1084100k^2 + 580k + 1\right), \qquad k \geq 99010.$$

*Proof.* A pinned MPFR/CAPD propagation embeds the exact launch curve on $[0.29, 0.290000001]$ in one correlated tripleton with $u$ frozen as a state coordinate. Before the exchange, every accepted full-step enclosure is checked for three positive mutual separations. The launch window verifies

$U < 2U_0$; subsequent windows verify the brake-exclusion disjunction of Corollary 4. Three algebraic changes of Levi–Civita chart use rigorous mean-value images.

Across the exchange, six $C^1$ Poincare maps project onto the transverse pair–13 sections

$$w_r = -\frac{3}{5}, -\frac{1}{2}, -\frac{2}{5}, -\frac{3}{10}, -\frac{1}{5}, 0.$$

For each map a convex mean-value domain contains the propagated family and its stored center, the validated section derivative transports the distinguished parameter generator, and an independent $C^0$ integration covers the complete incoming tube through the latest possible return. Thus section reconditioning does not omit any inter-section time. Every tube enclosure retains positive separation and excludes a brake.

For the terminal binary $\{2,3\}$ set $M = B+1$, $G = q_1 - C_{23}$, $\rho = |G|$, $h = |\dot{q}_3 - \dot{q}_2|^2/2 - M/r_{23}$, and

$$R = M/4, \quad d = \rho - R, \quad E_\rho = \frac{\dot{\rho}^2}{2} - \frac{A+B+1}{d}, \quad \Delta = \frac{A\sqrt{2MR}}{\sqrt{2E_\rho}\, d^2}.$$

These are precisely the quantities of Lemma 5; $h$ is transported as a regularized state variable. Hence the terminal check can use its enclosure without dividing by the small inner separation. The resulting terminal enclosure has

$$\begin{aligned} &t_* \in [4.30030837364, 4.30032429905], \\ &d > 1.8655, \qquad \dot{\rho} > 2.0484, \qquad E_\rho > 0.8223, \\ &h < -9.2745, \qquad -4 - h - \Delta > 5.0690. \end{aligned}$$

These are the strict hypotheses of the phase-robust binary–escaper theorem with binary $\{2,3\}$, escaper body 1, and $\eta = 4$. Hence after $t_*$ the inner pair either collides classically, ending the maximal classical solution, or body 1 escapes permanently; neither alternative permits a later brake. The second-brake lemma proves nonperiodicity. A separately compiled interval checker verifies the terminal inequalities on an outward-widened copy of the printed box. The source, dependency hash, and per-section tube audits are recorded in the repository certificate note (Appendix A). An archived second end-to-end run at 96-bit MPFR precision, tolerance $10^{-16}$, and Taylor order 26 reproduces all six section audits, 3344 main-flow steps, and the terminal margin.

The remaining extension is the union of 100 independently certified, adjacent closed tiles of exact width $10^{-10}$, beginning at 0.2900000001 and ending at 0.2900000101. Every requested interval is identical to its PASS interval, every consecutive pair shares its exact rational endpoint, and all 100 full logs are archived. The hardened base tile and the synchronized one-set theorem overlap this campaign, so their union is the complete closed interval stated above, with no topological or rounding gap. □

This continuum theorem covers a width of $1.01 \times 10^{-8}$. Larger parameter covers remain a separate validation problem; experimental wider propagations are not included in its conclusion.

For each fixed $u$ the separation bound is positive, and it is uniform on every compact parameter interval bounded away from $u = 0$. It degenerates in the skinny limit and is not a parameter-uniform Euclidean separation theorem on the whole open interval.

On any regular strict-maximum branch, the implicit-function theorem locally gives $t = t_k(u)$. The unresolved condition becomes the planar origin-avoidance problem

$$u \longmapsto \zeta(u, t_k(u)) \in \mathbb{C} \setminus \{0\}.$$

To turn this into a universal theorem one must also control branch creation, collision and escape endpoints, degenerate events, and the possible number of branches. The nominal dimension count alone supplies none of these facts.

Ordinary signed-event diagnostics rule out a simpler global shortcut. On the single (u=1/3) trajectory, maximum-event values of (zeta) occupy all four quadrants, and three early values have the origin in their convex hull. Thus no fixed real linear projection of $\zeta$ can be sign-definite over all maximum events. A successful obstruction must instead use branch-wise order or winding, a nonlinear barrier, or a state-dependent projection.

# 5 The equal-leg endpoint and its neighbors

**Theorem 8.** *At $u = \sqrt{2} - 1$, the classical solution has no positive second brake and reaches a binary or triple collision in finite time.*

*Proof.* Here $m_1 = m_2 = m = 1/\sqrt{2}$ and reflection symmetry persists. Write the base bodies as $(-x, y_b), (x, y_b)$ and the apex as $(0, y_a)$, with $h = y_a - y_b$. Then

$$\ddot{x} = -\frac{m}{4x^2} - \frac{x}{(x^2 + h^2)^{3/2}} < 0.$$

Starting with $x_0 = 1/2$ and $\dot{x}(0) = 0$ gives $\dot{x} < 0$, excluding a second brake. While $x > 0$, one has $\ddot{x} \le -m/(4x_0^2)$, whence

$$x(t) \le x_0 - \frac{m}{8x_0^2} t^2.$$

Thus collision occurs no later than $\sqrt{8x_0^3/m} = 2^{1/4}$. □

The endpoint is irrational, so this theorem does not settle an integer case. Its perturbation is singular because the collision unfolds into a close ordinary encounter when reflection symmetry is broken.

The endpoint itself has an exact collision and irrational parameter. Neither fact settles nearby rational triangles. Continuity in ordinary coordinates cannot carry a solution through a collision. The appropriate comparison uses the regularized endpoint trajectory while keeping the nearby physical trajectories classical.

Put $v = u/(\sqrt{2} - 1)$, $(m, n) = (A, B)$, $g = q_2 - q_1 = w^2$, and $G = q_3 - C_{12}$. With $dt = |w|^2 d\sigma$, write $z = w_\sigma$, $P = \dot{G}$, and $e = |\dot{g}|^2/2 - (m + n)/|g|$. The selected pair collision is $w = 0$; this is regular in the comparison equations, but it still terminates the classical trajectory at $v = 1$.

## 5.1 What makes the punctured-neighborhood argument work

There are three distinct ingredients, each needed for all-time exclusion. On the initial arc, exact torque inequalities and a validated first-syzygy map prove that nearby tied trajectories have no second brake. The word *syzygy* means that the three bodies are collinear. The normal launch tangent is fixed by the mass–shape relation; it is not a freely chosen perturbation.

At the first endpoint collision, an oriented $C^1$ Poincaré calculation gives $z_r < -4/5$ and $\partial_v w_i < -30$. Thus the map $(\sigma, v) \mapsto (w_r, w_i)$ has nonsingular derivative there. The nearby members miss that collision, with

$$\min r_{12} = \chi^2 (1 - v)^2 + o((1 - v)^2), \qquad \chi^2 > 900.$$

Their pair angular momentum changes sign across the encounter. Thus a launch torque-sign argument cannot be extended indiscriminately through later close passages.

The third ingredient continues through all four comparison collisions needed to reach an escape section. The detailed tangent equations, first-syzygy inequalities, and certificate constructions are retained in the technical companion; the following proof states how their strict bounds fit together.

**Theorem 9** (Punctured near-isosceles nonperiodicity)**.** *There exists $v_* < 1$ such that every real tied member with $v_* < v < 1$ is nonperiodic. It is collision-free and has no labelled brake through a fixed LC section; thereafter it either ends in a classical binary collision or has body 3 escape. Consequently the same conclusion holds for infinitely many primitive Pythagorean triples.*

*Proof.* The exact LC energy constraint is

$$2|z|^2 - (m+n) - e|w|^2 = 0.$$

On the symmetric endpoint orbit it gives $2|z|^2 = \sqrt{2}$ whenever $w = 0$, so every selected collision is transverse. Pinned oriented $C^1$ Poincare maps enclose four such zeros before $\sigma = 7$ and put the fifth strictly after 7. At each of the first four zeros, the tied normal derivative $(w_i)_v$ is bounded away from zero. Thus every Jacobian

$$\frac{\partial(w_r, w_i)}{\partial(\sigma, v)}$$

is nonsingular. The inverse-function theorem isolates all four zeros, while a 7000-step $C^0$ cover proves

$$r_{31}^2, r_{23}^2 > \frac{1}{500} \qquad (0 \le \sigma \le 7).$$

Compactness outside the four collision boxes now gives a punctured neighborhood in which every nearby tied orbit is collision-free through $\sigma = 7$.

At an ordinary point of this LC chart,

$$\dot{g} = \frac{2z}{\bar{w}}, \qquad \dot{G} = P.$$

Hence, after center-of-mass reduction, a labelled brake is equivalent to $z = P = 0$. The polynomial residual $\mathcal{R}_{\mathrm{LC}} = |z|^2 + |P|^2$ remains regular at comparison collisions. At the $\sigma = 1/2$ interface the cover gives $G_y > 1/10$, so the unique first syzygy lies later. Every complete interval tube from that overlapping pre-syzygy interface to $\sigma = 7$ satisfies $\mathcal{R}_{\mathrm{LC}} > 2$. Compact continuity transfers this inequality to the nearby classical orbits; the initial-arc result described above covers the initial segment.

At $\sigma = 7$, take $\eta = 4$ in the terminal binary–escaper theorem. The same interval cover proves, with $M = m + n = \sqrt{2}$, $R = M/4$ and $s = |G| - R$,

$$s > 1, \qquad \dot{\rho} > 2, \qquad \delta = \frac{\dot{\rho}^2}{2} - \frac{M+1}{s} > \frac{1}{20},$$

and

$$-4 - e - \frac{\sqrt{2MR}}{\sqrt{2\delta}\, s^2} > 2.$$

All inequalities are strict and therefore persist in a tied neighborhood. The terminal theorem says that each future trajectory either suffers an inner classical collision or escapes with uniformly positive outward radial speed. Neither possibility permits a later brake. Rational Euclid parameters are dense in the resulting open $u$ interval, giving infinitely many primitive triples. $\square$

The cutoff $v_*$ is existential. No numerical width is claimed for this endpoint neighborhood. An effective bound would identify a corresponding explicit range of integer triangles.

# 6 Thin triangles: close misses and escape windows

At the thin end of the family, the small mass $B = \epsilon$ is far from a tight pair of masses $A = \sqrt{1-\epsilon^2}$ and 1, initially separated by $\epsilon$. Two clocks then coexist: the inner pair moves on a time scale $\epsilon^{3/2}$, while the outer body returns on a time scale of order one. A single outer excursion contains an unbounded number of binary cycles as $\epsilon \to 0$. This is why excluding one early encounter does not settle all sufficiently thin triangles.

## 6.1 The first encounter does not collide

For $X = q_3 - q_1$, put $X = \epsilon x$ and $t = \epsilon^{3/2}\tau/\sqrt{A+1}$, and rotate so that $x(0) = 1$. The leading motion is radial Kepler fall. In Levi–Civita coordinates $x = z^2$, $d\tau = |z|^2 ds$, its solution is $z_0(s) = \cos(s/\sqrt{2})$, with collision at $s_c = \pi/\sqrt{2}$. The true parameter-dependent equations are analytic at this comparison collision. Their first transverse variation appears only at order five:

$$v_{ss} + \tfrac{1}{2}v = -\tfrac{3}{8}\cos^5(s/\sqrt{2}), \qquad v(0) = v_s(0) = 0, \qquad v(s_c) = -\frac{15\pi}{128}.$$

Consequently the physical first encounter misses collision by

$$r_{13,\min} = \frac{225\pi^2}{16384}\epsilon^{11}(1 + O(\epsilon)). \tag{4}$$

The proof uses the simple zero of $\operatorname{Re} z_0$ and the nonzero fifth-order imaginary displacement; away from that zero compact continuity keeps all pairs separated. Squaring the regularized displacement and restoring the factor $\epsilon$ gives the eleventh power. The symbolic coefficient calculation is recorded in the technical companion and `SKINNY_REGULARIZATION.md`.

The first passage is not an escape certificate. Its outer radial escape margin is $-2 + O(\epsilon)$, while the pair remains tightly bound. The subsequent return must be analyzed.

## 6.2 Match the binary clock to the returning body

Write $M = A + 1$, $N = M + \epsilon$, and $\rho_0^2 = 2A^2/M$. Replacing the inner pair by its combined point mass provides an explicit reference phase:

$$\Phi_{\mathrm{ref}}(\epsilon) = \frac{\pi}{\epsilon^{3/2}}\sqrt{\frac{M}{N}}\,\rho_0^{3/2} = \pi\epsilon^{-3/2} - \frac{\pi}{4}\epsilon^{-1/2} - \frac{15\pi}{32}\epsilon^{1/2} + O(\epsilon^{3/2}). \tag{5}$$

This is a reference clock, not an equality with the phase of the interacting system. The essential matching result compares *states*, rather than just an informal count of binary revolutions.

**Theorem 10** (Incoming-state matching)**.** *On a fixed sufficiently large scaled incoming section $\Sigma_K^-$, let $\Gamma_K^-(\chi)$ denote the incoming parabolic state of the equal-heavy-mass restricted problem, parameterized by its asymptotic mean-anomaly intercept $\chi$. As $\epsilon \to 0$, either a classical collision ends the exact tied orbit before it reaches the section, or its full regularized section state $S_\epsilon(K)$ satisfies*

$$\operatorname{dist}\big(S_\epsilon(K), \Gamma_K^-(\Phi_{\mathrm{ref}}(\epsilon))\big) = o(1).$$

*Argument and technical dependency.* Stop the outer infall at physical radius $\rho = \epsilon^\alpha$ with $0 < \alpha < 1/6$. Cancellation of the outer dipole gives a monopole-force remainder $O(\epsilon^2\rho^{-4})$. Inner-energy control, variation of the regularized oscillator, and comparison of its physical clock then give an incoming intercept error $O(\epsilon^{1/2-3\alpha}) = o(1)$. The remaining unbounded incoming tail has integrable quadrupole remainder $O(Y^{-4})$ in scaled radius $Y$. Integrating its energy, angular-momentum, and phase equations transports the full state to the fixed section. The intercept includes the finite quadrupole correction from infinity; using the raw monopole phase at the finite section would omit that term. The weighted tail bounds and uniform parameter estimates needed in this step are given in the technical companion's incoming-tail matching proof and `INCOMING_TAIL.md`. □

The corresponding restricted motion has a reversible parabolic separatrix. Its incoming and outgoing parabolic curves meet transversely. This claim requires pinning the phase: a whole-line variation without fixing the time translation has a decaying gauge mode and gives no transversality test. The finite certificate instead bounds the required Jacobi field on one regularized binary contraction. For its normalized terminal derivative $p$ it proves

$$p(\pi/2) > 1/125 \qquad \text{uniformly for } 14/5 \le v \le 4.$$

Here $v$ is a restricted launch-speed variable, unrelated to the near-isosceles parameter of Section 5. A parabolic stable-manifold argument then converts this strict sign into transverse intersection. The technical companion gives the compactified map and checks the hypotheses of the degenerate stable-manifold result [20]. The transversality supplies open escape-intercept arcs, as well as a returning side that remains to be classified globally.

### 6.3 Transfer open escape arcs back to integer triangles

Fix a compact subarc $J$ strictly inside a restricted escape window. Incoming-state matching and continuous dependence carry sufficiently small exact tied members with $\Phi_{\text{ref}}(\epsilon) \bmod 2\pi \in J$ to a finite outgoing section. There the binary specific energy and outer speed have the forms

$$e = -C/\epsilon + o(\epsilon^{-1}), \qquad \dot{\rho} = s\epsilon^{-1/2} + o(\epsilon^{-1/2}), \qquad C, s > 0,$$

with strict margins uniform on the compact arc. At a sufficiently distant scaled section, Lemma 5 applies. An earlier classical collision already excludes periodicity; every other member escapes or subsequently collides. This proves all-time exclusion on these windows, not just on the first excursion.

**Theorem 11** (Infinitely many thin-triangle intervals)**.** *There are infinitely many open intervals of real $u$ accumulating at zero on which the exact tied solution is nonperiodic. Each contains rational parameters and therefore primitive integer Pythagorean triples.*

*Proof.* Equation (5) gives $\Phi'_{\text{ref}}(\epsilon) \sim -(3\pi/2)\epsilon^{-5/2}$. The preimages of the interior of $J$ therefore include infinitely many disjoint intervals accumulating at zero. The uniform transfer above applies to every sufficiently small one. Finally $B(u)$ is a homeomorphism near zero and rational $u$ are dense. Different parameters in the fundamental interval give different primitive triples after scaling. □

**Theorem 12** (A positive-density explicit family)**.** *For a set of positive lower natural density of integers $n \ge 1$, the primitive triple*

$$(a_n, b_n, c_n) = (4n^2 - 1, 4n, 4n^2 + 1)$$

*has a nonperiodic Pythagorean–Burrau solution.*

*Proof.* Here $u_n = 1/(2n)$ and $B_n = 4n/(4n^2+1)$. The exact reference clock gives

$$F(x) := \frac{\Phi_{\mathrm{ref}}(B_x)}{2\pi} = \frac{(2x-1)^{3/2}(2x+1)(4x^2+1)^{3/4}}{32x^{5/2}},$$

with

$$F''(x) = \frac{3}{8}x^{-1/2} + \frac{1}{32}x^{-3/2} + O(x^{-5/2}).$$

The differentiated remainder follows from analyticity in $1/x$ after factoring out $x^{3/2}$. For each nonzero integer $h$, van der Corput's second-derivative estimate bounds the Weyl sum on $N < n \leq 2N$ by $O_h(N^{3/4})$. Dyadic summation and Weyl's criterion give equidistribution of $F(n)$ modulo one. Thus the compact arc $J$ selects indices of density $|J|/(2\pi) > 0$, and the escape transfer proves nonperiodicity for every sufficiently large selected index. Since $\gcd(4n, 4n^2-1) = 1$, the triples are primitive. This density concerns the displayed family of indices, not the set of all primitive Pythagorean triples ordered by hypotenuse. □

### 6.4 Why the remaining phases are harder

At boundaries of the restricted escape windows, returning orbits can pass near binary or triple collision. The detailed analysis in the companion constructs transverse planar collision boundaries and local finite-mass collision-or-escape regions. These are useful pieces of the eventual classification, but enumerating several boundaries does not enumerate all possible encounter sequences. Likewise, infinitely many real collision parameters do not imply that even one of them is rational, and a collision would in any case terminate the classical trajectory rather than supply a counterexample. The main unresolved thin-triangle problem is a uniform classification of the remaining returning components.

## 7 Torque history and the remaining analytic obstruction

The finite-cover strategy can prove particular intervals, but an analytic constraint on all reachable trajectories would reach further. Pair torques provide such a candidate constraint because they remember the entire motion since release. The identities in this section are exact; their proposed global sign consequence remains unproved.

### 7.1 What a second brake must cancel

Let $d_{ij} = q_j - q_i$, $r_{ij} = |d_{ij}|$, and $\Delta_2 = (q_2 - q_1) \times (q_3 - q_1)$ be twice the signed area. Define the specific pair angular momenta $\ell_{ij} = d_{ij} \times \dot{d}_{ij}$, using the cyclic pairs $12, 23, 31$. Direct differentiation gives

$$\begin{aligned} \dot{\ell}_{12} &= m_3\Delta_2(r_{23}^{-3} - r_{31}^{-3}), \\ \dot{\ell}_{23} &= m_1\Delta_2(r_{31}^{-3} - r_{12}^{-3}), \\ \dot{\ell}_{31} &= m_2\Delta_2(r_{12}^{-3} - r_{23}^{-3}). \end{aligned} \tag{6}$$

Their derivatives have signs $(-, +, -)$ at launch in the strict fundamental interval. Since every $\ell_{ij}$ vanishes at a brake, each right-hand side must integrate to zero before a second brake. Its initial strict sign must therefore be reversed somewhere. Before the first syzygy, this requires reversals of all three initial side comparisons, at possibly different times. It does not prove that those reversals cannot occur.

Integrating Lagrange–Jacobi between two brakes also gives

$$\int_0^\tau U\,dt = 2U_0\tau, \qquad \int_0^\tau K\,dt = U_0\tau.$$

These useful necessary conditions are not pointwise monotonicity laws. The time-dependent torque signs and the change of area orientation at syzygies are precisely what a global argument must control.

### 7.2 Reduce the memory to a weighted average

On a positive-area ordered arc $r_{12} > r_{23} > r_{31}$, put $R = r_{12}$, $x = r_{23}/R$, $y = r_{31}/R$, and define

$$F = \Delta_2(r_{31}^{-3} - R^{-3}) > 0, \qquad G = \Delta_2(r_{23}^{-3} - R^{-3}) > 0, \qquad k = \frac{G}{F} = \frac{y^3(1-x^3)}{x^3(1-y^3)}.$$

Integrating the torques from release gives

$$\eta = \frac{\int_0^t G\,d\tau}{\int_0^t F\,d\tau} = \frac{m_1}{m_2}\frac{-\ell_{31}}{\ell_{23}}, \qquad \dot\eta = \frac{F}{\int_0^t F\,d\tau}(k-\eta). \tag{7}$$

Thus $\eta$ is a weighted average of the preceding values of $k$. If $k$ strictly decreases, then $k(t) < \eta(t) < k(0)$. What remains is to prove the required decrease for states actually reached from the brake.

Use shape time $ds = dt/R^{3/2}$ and write $W = \ell_{23}/\sqrt{R}$, $\delta = \Delta_2/R^2$. Eliminating the two shape velocities gives the rational identity

$$(\log k)_s = \frac{W}{\delta}\mathcal{A}(u,x,y)(\eta - h(u,x,y)), \qquad \mathcal{A} > 0, \quad h > k. \tag{8}$$

Here $h$ is the torque threshold, unrelated to the binary energy called $h$ in the middle-interval certificate. Explicit polynomials defining $h$ and $\mathcal{A}$, and their exact Bernstein sign checks, are given in the technical companion. Their role is to replace a velocity-sign problem by the scalar condition $\eta < h$. This condition is established on the first arc near the equal-leg endpoint, but not on the entire fundamental interval.

### 7.3 A corrected conditional sign-change argument

Define the gap, lag, and threshold distance

$$d = h - k > 0, \qquad e = \eta - k, \qquad w = h - \eta = d - e,$$

and the positive coefficients

$$\lambda = \frac{m_1\delta(y^{-3}-1)}{W}, \qquad c = \frac{kW\mathcal{A}}{\delta}.$$

Equations (7)–(8) give

$$e_s + (\lambda + c)e = cd, \qquad w_s + (\lambda + c)w = q, \qquad q = d_s + \lambda d. \tag{9}$$

The notation $q$ here denotes a scalar forcing, not a body position. For $s$ strictly before syzygy, the launch condition has the improper integral representation

$$e(s) = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^s c(\tau)d(\tau)\exp\left[-\int_\tau^s(\lambda + c)\,d\xi\right]d\tau.$$

At launch $W$ has a simple zero, $\lambda \sim 1/s$, and $e(0) = 0$ after analytic extension, so the homogeneous launch contribution tends to zero. One must not define the integrating factor by integrating from zero. For a fixed interior reference point $s_0 > 0$, the legitimate definition is

$$M(s) = \exp \int_{s_0}^{s} (\lambda + c)\, d\xi, \qquad (Mw)_s = Mq.$$

There is a second endpoint issue. At the first ordered syzygy $s = S$, collinear momentum constraints force $w(S) = 0$. But $\delta \to 0$ there, and $c = kW\mathcal{A}/\delta$ can diverge. It does not follow that $Mw$ tends to zero. An earlier draft used exactly this unjustified implication in a proposed one-sign-change proof.

**Lemma 13** (Sufficient sign-change criterion with terminal control)**.** *Assume $w > 0$ near launch, $q$ is positive up to one interior sign change and negative thereafter, and*

$$\liminf_{s \uparrow S} M(s)w(s) \geq 0. \tag{10}$$

*Then $w > 0$ throughout $0 < s < S$.*

*Proof.* Start at any sufficiently small positive time where $w > 0$. Since $M > 0$, the product $Mw$ increases until the switch and then strictly decreases. If it reached zero before $S$, its terminal lower limit would be negative, contrary to (10). □

The missing hypothesis cannot be omitted from this scalar argument. On $0 < s < 1$ take

$$a = \frac{1}{s} + \frac{1}{1-s}, \qquad w = (1-s)(1+s-3s^2), \qquad q = (1-s)\left(\frac{1}{s} + 2 - 9s\right).$$

Then $w_s + aw = q$, $w(0) = 1$, and $w(1) = 0$. The forcing changes from positive to negative exactly once, at $(1+\sqrt{10})/9$, yet $w$ becomes negative after $(1+\sqrt{13})/6$. Up to a positive constant, $M = s/(1-s)$ and $Mw \to -1$. This example refutes the previously stated scalar inference. It does not construct a three-body trajectory, and it does not refute the Pythagorean conjecture. A valid global use of the lag equation needs a reachable terminal bound such as (10), or another argument that controls the sign without that implication.

### 7.4 Why present shape and energy are insufficient

A stronger route would impose the desired torque or distance-order barrier on every state with the correct energy and angular momentum. Exact ambient counterexamples rule out this shortcut. One is particularly transparent. For masses $(m, n, 1) = (4/5, 3/5, 1)$, consider an ordered collinear state with

$$(r_{12}, r_{23}, r_{31}) = (1, 697/700, 3/700), \qquad \eta = 419/32759, \qquad \ell_{23}^2 = \frac{476958362267}{2867548600}.$$

Choose zero radial velocities and the transverse velocities fixed by these angular momenta and $P = L = 0$. Exact arithmetic gives $H = -U_0$, pair angular-momentum signs $(-,+,-)$, and first-crossing area orientation $\dot{\Delta}_2/\ell_{23} = -292607/131036 < 0$, but

$$(r_{23} - r_{31})'' = -\frac{96354167469624287}{1827657180810} < 0.$$

This state is not asserted reachable from the tied initial brake. It shows that energy, collinear geometry, crossing direction, and pair angular-momentum signs do not suffice to control the second distance gap. The main analytic target is therefore a bound on *reachable histories*. Further static witnesses and the exact amplitude inequalities are retained in the technical companion, rather than repeated here.

## 8 What remains to prove

The stronger real-parameter conjecture would follow by excluding every positive zero of the brake residual for all $0 < u \le \sqrt{2}-1$. On a regular strict-maximum branch, the implicit-function theorem gives an event time $t_k(u)$, and the remaining condition is

$$\zeta(u, t_k(u)) \neq 0.$$

An argument restricted to regular branches must also account for their creation and termination, degenerate events, collisions, escape, and possible unbounded encounter counts. At a Jacobi degeneracy the Hopf residual replaces the complex chart. None of these obligations follows from counting equations and parameters.

The partial results address different parts of that task:

- The middle-interval certificate covers every time up to a terminal escape condition, but only on its stated narrow parameter interval.
- The near-isosceles argument proves a whole punctured neighborhood, but currently gives no explicit cutoff.
- The thin-triangle argument gives open escape windows and an explicit positive-density subfamily, while leaving returning phases to classify.
- The torque-history identities reduce part of the first-arc problem to scalar quantities; a global reachable bound, including the singular terminal behavior, is still missing.

If a real periodic point is found instead, the integer conjecture requires an additional exact arithmetic step. A real zero need not have rational $u$. Conversely, a countable dense set of rational parameters cannot be dismissed by a generic dimension count, a measure-zero claim, or a finite scan. The next decisive advance must either supply a global exclusion mechanism or isolate a genuinely Pythagorean periodic orbit with the necessary exact and collision-free certification.

## A Proof dependencies and reproducibility

The expanded analytic calculations and certificate arguments are in the technical companion, `technical-details.pdf`. This research draft includes computer-assisted results. The repository notes below identify verifier sources, assumptions, pinned dependencies, and archived outputs. The September 2026 manuscript review checks the logical assembly and runs selected exact regressions; it is not a fresh independent replay of every interval certificate. Ordinary shooting data are not promoted to interval theorems by appearing in the same document.

| Result or ingredient | Repository proof/certificate entry point |
|---|---|
| Second-brake and maximum reduction | `docs/FABLE_EVENT_REDUCTION.md` |
| Terminal escape | `docs/ESCAPE_CRITERIA.md`; `docs/FABLE_MIDDLE_ESCAPE.md` |
| First maximum, [.29, .29002] | `docs/MIDDLE_FIRST_MAXIMUM_INTERVAL.md` |
| All-time middle interval | `docs/FABLE_MIDDLE_INTERVAL_NONPERIODICITY.md` |
| Near-equal-leg interval | `docs/ISOSCELES_SYZYGY_THRESHOLD.md`; `docs/ENDPOINTS.md` |
| First thin-triangle miss | `docs/SKINNY_REGULARIZATION.md` |
| Incoming-state matching | `docs/INCOMING_TAIL.md` |
| Explicit positive-density family | `docs/EXPLICIT_SKINNY_FAMILY.md` |
| Torque identities and correction | `docs/EXACT_REDUCTIONS.md`; `docs/FORCED_LAG_ENDPOINT_CORRECTION.md` |
| Counterexample search | `docs/PERIODIC_CONSTRUCTION_REPORT_2026-09-05.md` |
| Review and historical sources | `docs/MANUSCRIPT_REVIEW_2026-09-05.md`; `docs/PYTHAGOREAN_HISTORY_SOURCES.md` |

The original fan image is reproduced without changing its pixels; its metadata are archived in `paper/figures/fan-provenance.json`. The script `paper/make_figures.py` generates the two new vector figures from the exact parameterization and the campaign data. Build instructions and the preserved pre-review source and PDF are described in `paper/README.md`.

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